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WAITING FOR A GAP IN TRAFFIC

by

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ABSTRACT

This paper considers the waiting-time of a driver on a secondary road who wishes to enter or cross a primary traffic stream, given that he will do so only when the spacing to the next vehicle is at least as great as some fixed critical gap. The initial waiting-time distribution and its first two moments are obtained under the assumption of arbitrary starting-up and headway distributions of the primary traffic stream. Under the assumption that only one car may enter the intersection per gap, the distribution of successive waiting times (the inter-gap distribution) and its moments are obtained. The counting distribution of the passing of gaps is then used to describe the numbers of cars which leave the secondary road during a fixed interval of time. Various asymptotic and limiting results are demonstrated, together with the special results obtained when the traffic is Poisson. The necessary modifications for other gap criteria are also indicated.

WAITING FOR A GAP IN TRAFFIC*

Introduction

The phenomenon of waiting to enter or cross a stream of traffic is a familiar one to all of us. Instead of the usual queueing situation in which the delay is due to the completion of several service tasks of variable length, the delay in entering a stream of traffic is caused by the successive passage of a varying number of inter-car spacings, each of which is not large enough to allow entry. Thus, in this problem, the main stochastic sequence (the stream of traffic) acts like a sieve, allowing one to enter only when a spacing greater than or equal to some critical length appears. This critical length depends upon the velocity of the main stream, the reaction time of the waiting driver, his acceleration ability, the geometry of the intersection, and so forth.

In this paper we shall describe a simple recurrent event model which includes many of the features of the realistic situation. In addition to deriving the first passage and recurrence distributions of the waiting time, we show the first two moments of these distributions and consider their asymptotic forms. By deriving the counting distribution of 'large-enough' gaps, the emptying of an infinite queue on the secondary road is also described. Explicit results are presented for the case of Poisson traffic, and

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* This paper was originally presented at the Second Western Joint Meeting of the Operations Research Society of America and Institute of Management Sciences, Monterey, California, April 15-16, 1960, while the author was an employee of Broadview Research Corporation. At that time only a summary of the formulas presented here was distributed [5]. Several revisions in notation have been made in this complete version of the paper to conform with a forthcoming report by Oliver [11], and a new section has been added to demonstrate the effect of variable gap criteria.

the necessary modifications for other gap criteria are presented.

Besides its use in describing delays of secondary traffic, this model has applications in scheduling traffic-actuated signal lights, and in describing car passing and left-turn queueing phenomena. Little [7][8] has applied related results to the location of retail stores near an intersection, under the assumption that customers tend to prefer stores with a minimum of delay in entering and leaving the parking area.

The Model

Consider the idealized situation of Figure 1, where successive cars in a single lane of traffic pass the intersection at times $\tau_1, \tau_2, \tau_n, \ldots$, measured from the measurement origin $\tau_o = 0$. The possible effect of upstream dynamics, or the "bluffing" of the waiting driver, are ignored.

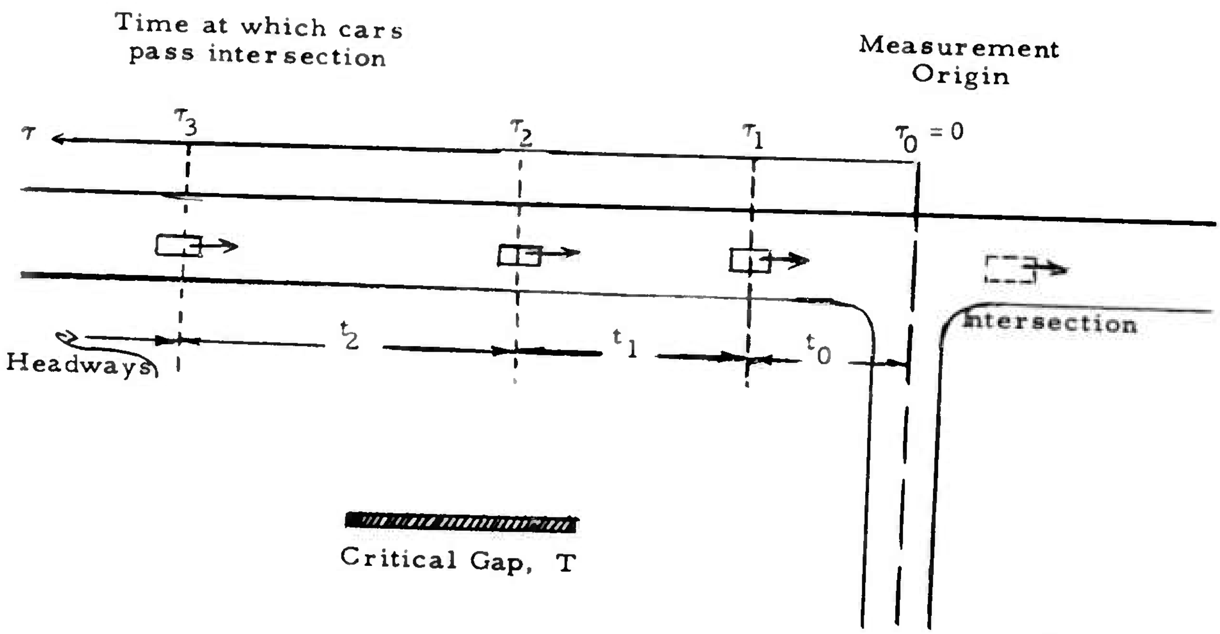

Figure 1 -- Model Description

We neglect the dimensions of passing cars and assume that the instants of arrival at the intersection are generated by a renewal process, i.e., the non-negative time spacings

$$t_0 = \tau_1 - 0, \quad t_1 = \tau_2 - \tau_1, \quad t_2 = \tau_3 - \tau_2, \quad \ldots, \quad t_n = \tau_{n+1} - \tau_n, \quad \ldots .$$

are random variables from a known interevent distribution. The time spacings $t_1, t_2, \ldots, t_n, \ldots$ are referred to as headways in traffic studies.

In order to retain generality about the measurement origin in the sequel, we shall assume that t_0 is a sample from a starting-up distribution density $d(t)$, and that $t_1, t_2, \ldots, t_n, \ldots$ are independent samples from a headway distribution density $a(t)$.

It is assumed that the driver on the secondary road will enter or cross the main stream of traffic if and only if the time interval, until the next car passes, is greater than or equal to some critical gap size, T. This critical gap is assumed to remain the same for every driver, and not subject to change because of impatience.

If a car on the secondary road begins to wait at time zero, then we define his waiting time, w, as

$$\begin{array}{ll}
w = 0 & \text{if } t_0 \geq T \\
w = t_0 & \text{if } t_0 \leq T,\ t_1 \geq T \\
w = t_0 + t_1 & \text{if } t_0 < T,\ t_1 < T,\ t_2 \geq T \\
\vdots & \qquad \vdots \\
w = \sum\limits_{i=0}^{n} t_i & \text{if } t_0 < T,\ t_1 < T,\ t_2 < T,\ \ldots\ t_n < T,\ t_{n+1} \geq T\ .
\end{array} \tag{1}$$

Thus, the wait will be zero if the driver is able to enter the intersection before the first car arrives; if this initial headway is not large enough, we define his waiting time as the time until the beginning of the first gap, i.e., the beginning of the first inter-car headway not less than T.

Simple models of this type were first considered by Garwood [3], Raff [12], and Tanner [13], in order to derive the waiting-time distribution for Poisson traffic. Among their results were the formulas of Equations (22), (23), and (24). Mayne [9] was the first person to consider arbitrary headway distributions under the assumption of beginning measurements "at random," obtaining results which are equivalent to Equation (16), and its associated moments.

Since this paper was presented [5], Weiss and Maradudin [14] have presented a generalization of this model, in which a certain probability of entering the intersection is associated with each inter-vehicle headway. The results obtained, while more complicated, permit a more realistic description of driver acceptance criteria. Results which are generalizations of (16), (36), and (37), together with their moments are obtained. Weiss and Maradudin have also made a preliminary formulation of the problem of correlated headways. In the last section of this paper is indicated a simple manner in which our results can be generalized to include variable gap criteria.

R. M. Oliver [11] has taken all of the results for a critical gap criteria of $\geq$ T, unified the notation and formulas, and presented new results on the distributions of blocked and unblocked periods. The emphasis of his paper is upon the gap and block-producing mechanism of the primary traffic stream, with the waiting-time problem treated as a special secondary process. Because of the importance of this approach, several revisions in notation have been made in the present paper.

Notation

The following notation will be used: Let f(t) be any distribution density, which may possibly have discrete components (represented by impulses or Dirac-delta functions).

Then the tail cumulative distribution function will be represented by F(t).

$$F(t) = \int_{t-}^{\infty} f(x)\, dx \tag{2}$$

The truncated distribution density, f(t;T) is given by:

$$\begin{aligned} f(t;T) &= f(t) \qquad 0 \leq t < T \\ &= 0 \qquad T \leq t \end{aligned} \tag{3}$$

Notice that the total area under f(t;T) is just 1 - F(T).

We shall use a tilde and the argument s to denote a LaPlace transform, viz.

$$\tilde{f}(s) = \mathcal{L}\,[\, f(t)\,] = \int_0^{\infty} f(t)\ \epsilon^{-st}\, dt \tag{4}$$

We shall use the convolution notation

$$f^{n*}(t) = f^{(n-1)*}(t) * f(t) \qquad n = 2,\ 3,\ \ldots \tag{5}$$

where the convolution operation $*$ is defined as

$$f(t) * g(t) = \int_0^{t} f(x)\ g(t-x)\, dx \tag{6}$$

and for convenience we define

$$f^{1*}(t) = f(t) \tag{7}$$

We shall use ν_f and σ_f^2 to denote the mean and variance of the distribution density $f(t)$; where no confusion results we shall eliminate the subscript on the moments of the mainstream headway distribution, $a(t)$. If the mean is conditional upon the values of the random variable being less than some value T, we add that value as an argument; i.e., $\nu_f(T)$ is the mean of the normalized truncated distribution $[1 - F(T)]^{-1} f(t;T)$.

The Distribution of Initial Wait

The measurement origin and the successive passage of automobiles through the intersection presents a series of Bernoulli trials, with probabilities $D(T)$, $A(T)$, $A(T)$, ... of success on the 0^{th}, 1^{st}, 2^{nd}, ... trials. Hence the probability of the initial waiting time terminating at the n^{th} trial (i.e., at time τ_n) is just $[1 - D(T)] [A(T)] [1 - A(T)]^{n-1}$ $(n = 1, 2, \ldots)$, and probability $D(T)$ of terminating at the origin.

If we use $q(t)$ for the distribution density of initial wait we see that it has a discrete component at zero, since

$$Pr \left\{ w = 0 \right\} = D(T).$$

If, on the other hand, the waiting time terminates on the first trial, the initial headway, t_0, must have been less than T, and hence the (normalized) distribution of waiting time must be $d(t;T) \; [1 - D(T)]^{-1}$.

Similarly, if the waiting time terminates on the n^{th} trial, the headways $t_0, t_1, t_2, \ldots t_{n-1}$ must all be less than T, and the normalized distribution of waiting time is

$$\left\{ d(t;T) \; [1 - D(T)]^{-1} \right\} * \left\{ a^{(n-1)*}(t;T) \; [1 - A(T)]^{n-1} \right\} \; (n \geq 2)$$

since $w = t_0 + t_1 + t_2 + \ldots . \; t_{n-1}$.

If these normalized waiting time distributions are multiplied by the probabilities of success on the corresponding trials and summed, we obtain:

$$q(t) = D(T)\ \delta(t) \quad + \quad d(t;T) * \left[1 + \sum_{j=1}^{\infty} a^{j*}(t;T) \right] A(T) \tag{8}$$

where $\delta(t)$ is the Dirac-delta function. Thus equation (8) is the distribution density of initial wait at the intersection until a gap at least as large as T appears in the main stream of traffic.

By taking transforms, and assuming series summability, we find the more compact expression:

$$\tilde{q}(s) = D(T) + \frac{\tilde{d}(s;T)\ A(T)}{1 - \tilde{a}(s;T)} \ . \tag{9}$$

The advantage of this representation is that the moments of $q(t)$ are available through differentiation of the transform.

For example,

$$\lim_{s \to 0+} \tilde{q}(s) = \int_0^{\infty} q(t)\, dt = D(T) + \frac{(1 - D(T))\ A(T)}{A(T)} = 1$$

as it should.

The mean initial wait is

$$\nu_q = \lim_{s \to 0+} \left[- \frac{d\tilde{q}(s)}{ds} \right] = \int_0^T t\ d(t)\ dt \quad + [1-D(T)]\ [A(T)]^{-1} \int_0^T t\ a(t)\ dt \tag{10}$$

We may rewrite this in a more suggestive form as

$$\nu_q = [1 - D(T)] \left[\nu_d(T) + \frac{1 - A(T)}{A(T)} \nu(T)\right],$$

where $[1 - A(T)]/A(T)$ is mean number of cars passing after the first one, given that at least one car passes before the wait is ended.

The mean squared initial wait is

$$\sigma_q^2 + (\nu_q)^2 = \lim_{s \to 0+} \left[\frac{d^2\tilde{q}(s)}{ds}\right] = \int_0^T t^2 d(t)\,dt + \frac{2}{A(T)} \left[\int_0^T t\, d(t)\,dt\right]\left[\int_0^T t\, a(t)\,dt\right]$$
$$+ \left[\frac{1-D(T)}{A(T)}\right] \int_0^T t^2\, a(t)\,dt + \frac{2[1-D(T)]}{[A(T)]^2}\left[\int_0^T t\, a(t)\,dt\right]^2 . \tag{11}$$

Different Measurement Origins

The choice of the starting-up density, $d(t)$, depends upon the relationship of the origin of measurement relative to the traffic stream.

In the simplest case, waiting time begins just after another car has passed by, and $d(t) = a(t)$. We shall use the special notation $b(t)$ for the wait distribution in this case. We find from (9) the transform of the waiting distribution to be:

$$\tilde{b}(s) = \frac{A(T)}{1 - \tilde{a}(s;T)} \tag{12}$$

which has a mean

$$\nu_b = \frac{\int_0^T t\; a(t)\,dt}{A(T)} = \left[\frac{1-A(T)}{A(T)}\right] \nu(T) , \tag{13}$$

and a variance

$$\sigma_b^2 = \int_0^T \frac{t^2 \, a(t)\, dt}{A(T)} + (\nu_b)^2 \quad . \tag{14}$$

Another important case is that in which the waiting time begins at a random moment relative to the main stream of traffic. One can show (see, for example, Ref. [6]) that the correct starting-up density to use is:

$$d(t) = u(t) = \nu^{-1} A(t) \quad . \tag{15}$$

For starting to wait "at random", we shall use the special notation $w(t)$ for the initial waiting time distribution density. We have

$$\tilde{w}(s) = U(T) + \frac{\tilde{u}(s;T)\, A(T)}{1 - \tilde{a}(s;T)} \quad , \tag{16}$$

with the starting distributions defined through (15). Expressions for the moments of $w(t)$ can be developed completely in terms of moments of the headway distribution, but are not particularly revealing.

Another possibility occurs when the measurement origin is a fixed time τ after the time at which the <u>last previous</u> car passed the intersection. In this case, one should use

$$d(t) = a(t + \tau)/A(\tau) \qquad t \geq 0 \tag{17}$$

Finally, it is sometimes a convenience to begin measurements at a fixed time τ after the time at which <u>some previous</u> car passed the intersection, no measurements having been taken in the interim period. It can be shown in this case [6] that the starting-up density

$$d(t) = a(t \mid \tau) \tag{18}$$

has the double LaPlace transform

$$\tilde{\tilde{a}}(s_1 \mid s_2) = \frac{\tilde{a}(s_1) - \tilde{a}(s_2)}{(s_2 - s_1)[1 - a(s_2)]} \tag{19}$$

Another possible starting-up distribution will be considered in the section on inter-gap spacings.

Poisson Traffic

The special case of Poisson traffic is of interest, both theoretically and practically, since headway distributions in freely flowing, sparse traffic are often found to be close to exponential [1]. Furthermore, it turns out that for Poisson traffic all of the starting-up densities (15), (17), and (19) are equivalent to assuming $d(t) = a(t)$! Thus, for all of these measurement origins, we may obtain the distribution of initial waiting time from (12), and from the Poisson inter-event density:

$$a(t) = \lambda \exp(-\lambda t) = \lambda A(t) \qquad (t \geq 0) \tag{20}$$
$$\nu = \sigma^2 = \lambda^{-1}$$

The transform of the waiting-time distribution is:

$$\tilde{b}(s) = \tilde{w}(s) = \frac{(s+\lambda)\,\epsilon^{-\lambda T}}{s + \lambda\epsilon^{-\lambda T}\epsilon^{-sT}} \tag{21}$$

which gives

$$b(t) = w(t) = \epsilon^{-\lambda T}\,\delta(t) \tag{22}$$
$$+ \lambda\epsilon^{-\lambda T} \qquad 0 \leq t \leq \infty$$

$$- \lambda\epsilon^{-2\lambda T}\left[1 + \frac{\lambda(t-T)}{1!}\right] \qquad T \leq t \leq \infty$$

$$+ \lambda\epsilon^{-3\lambda T}\left[\frac{\lambda(t-2T)}{1!} + \frac{\lambda^2(t-2T)^2}{2!}\right] \qquad 2T \leq t \leq \infty$$

$$- \lambda\epsilon^{-4\lambda T}\left[\frac{\lambda^2(t-3T)^2}{2!} + \frac{\lambda^3(t-3T)^3}{3!}\right] \qquad 3T \leq T \leq \infty \quad .$$

$$\cdots\cdots \qquad\qquad \cdots\cdots$$

Each successive term is added only in the range indicated on the right. This distribution was first obtained by Raff [12] and Tanner [13], and later obtained in different ways by other investigators, [4] and [7].

The continuous portion of (22) is shown in Figure 2 for $\lambda T = \frac{1}{2}$, 1, and 2. It consists of piecewise-continuous curves of length T, which are successively constant, straight-line, parabolic, cubic, etc.

The mean initial wait is given by (13) as:

$$\nu_b = \nu_w = \lambda^{-1}(\epsilon^{\lambda T} - 1 - \lambda T) \tag{23}$$

$$\approx \lambda^{-1}\left[\frac{\lambda^2 T^2}{2!} + \frac{\lambda^3 T^3}{3!} + \cdots\right] \qquad T \to 0$$

$$\approx \lambda^{-1}\epsilon^{\lambda T} \qquad T \to \infty$$

This formula was first obtained by Adams [1]. Thus, in Poisson traffic, the mean wait increases at least as the square of the critical gap, T, and increases at least linearly with the mean traffic flow rate, λ. For large values of T, the mean wait increases exponentially.

Continuous Portion of Probability Density of Initial Wait, $b(t) = w(t)$ in Poisson Traffic of Intensity λ

For	$\lambda T = 0.5$	$Pr\ t=0 = 0.607$,	$\lambda \nu_b = 0.149$
	$\lambda T = 1.0$	$Pr\ t=0 = 0.368$,	$\lambda \nu_b = 0.718$
	$\lambda T = 2.0$	$Pr\ t=0 = 0.135$,	$\lambda \nu_b = 4.389$

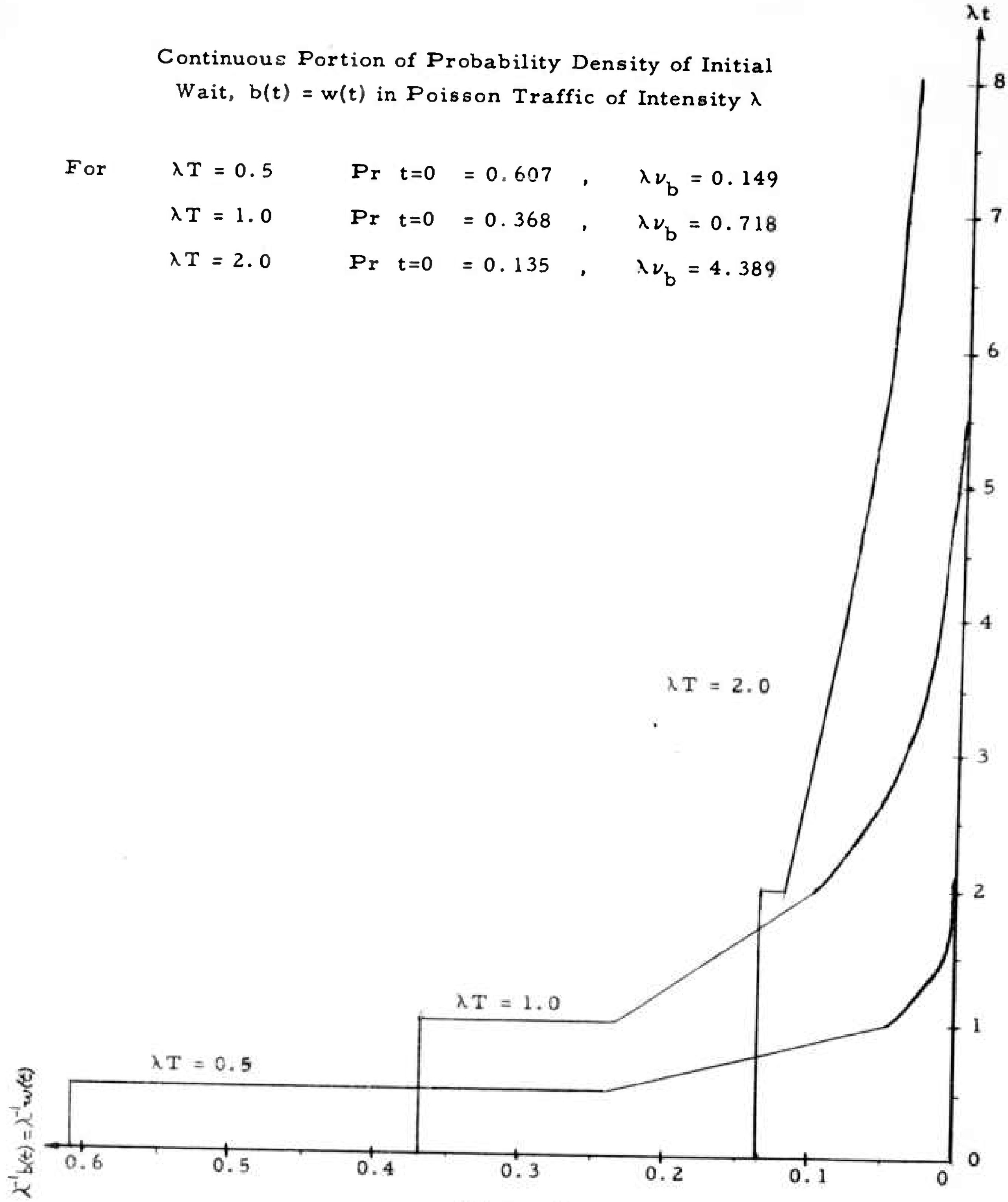

Figure 2

The variance of mean wait in Poisson traffic is

$$\sigma_b^2 = \sigma_w^2 = \lambda^{-2}\left[\epsilon^{2\lambda T} - 1 - 2\lambda T\epsilon^{\lambda T}\right] \tag{24}$$

$$\approx \lambda^{-2}\left[\frac{2\lambda^3T^3}{3!} + \frac{8\lambda^4T^4}{4!} + \dots\right] \qquad T \to 0$$

$$\approx \lambda^{-2}\epsilon^{2\lambda T} \qquad T \to \infty$$

and it increases at least as the cube of the critical gap, T.

Limiting Results

The dependences on T just described turn out to be true for more general traffic headway distributions.

Assume that a(t) and d(t) have expansions about the origin of the form

$$a(t) = \sum_{j=0}^{\infty} a_j \frac{t^j}{j!} \qquad d(t) = \sum_{j=0}^{\infty} d_j \frac{t^j}{j!} . \tag{25}$$

From equations (10) and (11), we find, after some algebra

$$\lim_{T\to 0} \nu_q = d_0\frac{T^2}{2!} + \left[2d_1 + 3d_0a_0\right]\frac{T^3}{3!} + \left[3d_2 + 6d_1a_0 + 8d_0a_1 + 12d_0a_0^2\right]\frac{T^4}{4!} + \dots \tag{26}$$

and

$$\lim_{T\to 0} \sigma_q^2 = 2d_0\frac{T^3}{3!} + \left[6d_1 + 20d_0a_0 - 6d_0^2\right]\frac{T^4}{4!} + \left[12d_2 + 60d_1a_0 - 40d_1d_0 + 70d_0a_1 - 60d_0^2a_0 + 160d_0a_0^2\right]\frac{T^5}{5!} + \dots \tag{27}$$

In the special case of starting to wait just after a car has passed

$$\lim_{T \to 0} \nu_b = a_0 \frac{T^2}{2!} + \left[2a_1 + 3a_0^2\right] \frac{T^3}{3!} + \left[3a_2 + 14a_1a_0 + 12a_0^3\right] \frac{T^4}{4!} + \dots \tag{28}$$

$$\lim_{T \to 0} \sigma_b^2 = 2a_0 \frac{T^3}{3!} + \left[6a_1 + 14a_0^2\right] \frac{T^4}{4!} + \left[12a_2 + 90a_1a_0 + 100a_0^3\right] \frac{T^5}{5!} + \dots \tag{29}$$

If measurements are begun at random relative to the main stream process,

$$\lim_{T \to 0} \nu_w = \nu^{-1} \left[\frac{T^2}{2!} + a_0 \frac{T^3}{3!} + [5a_1 + 6a_0^2] \frac{T^4}{4!} + \dots \right] \tag{30}$$

and

$$\lim_{T \to 0} \sigma_w^2 = \nu^{-1} \left[2 \frac{T^3}{3!} + [14a_0 - 6\nu^{-1}] \frac{T^4}{4!} + [58a_1 + 100a_0 - 20a_0\nu^{-1}] \frac{T^5}{5!} + \dots \right] \tag{31}$$

Thus, in all cases, the mean wait increases at least as the square of T and the variance increases at least as the cube of T.

We also note from (26) and (27) the curious result

$$\lim_{T \to 0} \sigma_q^2 = \alpha [\nu_q T] \tag{32}$$

$$\begin{aligned} \text{where} \quad \alpha &= 2/3 \quad d_0 \neq 0 \\ &= 3/4 \quad d_0 = 0,\ d_1 \neq 0 \\ &\ \ \vdots \\ &= \frac{2+n}{3+n} \quad d_0 = 0,\ d_1 = 0,\ \dots\ d_{n-1} = 0,\ d_n \neq 0 \end{aligned}$$

Relationships similar to (26) and (27) can also be for "stuttering" distributions [6], in which cars can arrive in geometrically distributed groups.

For very large values of the critical gap, T, the wait increases without bound as A(T) vanishes. Asymptotic limits can be found directly from (10) and (11), giving,

$$\lim_{T \to \infty} \nu_q = \nu_d + \nu A^{-1}(T) \approx \nu A^{-1}(T) \tag{33}$$

$$\lim_{T \to \infty} \sigma_q^2 \approx \sigma_d^2 + [\sigma^2 + \nu^2] A^{-1}(T) + \nu^2 A^{-2}(T) \approx \nu^2 A^{-2}(T) \tag{34}$$

In the limit, (33) gives the intuitive expression for mean wait, since A(T) is the average fraction of gaps which are large enough to permit entering or crossing the main stream.

Notice that the coefficient of variation

$$k_q^2 = \sigma_q^2 / (\nu_q)^2 \tag{35}$$

approaches unity for large values of T, even though the initial wait distribution is not exponential.

The Distribution of Spacings Between Gaps

Having derived the initial waiting time distribution, we are interested in finding the waiting time for successive cars which are lined up in a queue on the side road.

The simplest situation occurs when each car on the secondary road can enter or cross the main stream of traffic when triggered by the passing of a car on the main stream; that is, even if a headway in the main stream is large enough for 2, 3, 4, ... cars to enter the intersection, only one car is allowed. This situation might arise, for example, when visibility was limited to an equivalent distance of less than 2T, and successive cars waited until a regeneration point of the main process to make certain that there was headway enough to enter. This model is also a very good approximation for the case where

$A(2T) << A(T)$, as in heavy traffic.*

To obtain the distribution of wait from the beginning of one gap to the beginning of the next one, we use (9), with the starting-up distribution equal to the distribution of a headway which is a gap, i.e.,

$$d(t) = g(t) = \frac{a(t) - a(t;T)}{A(T)} \tag{36}$$

Using the special notation, $c(t)$, for the inter-gap spacing, we find its transform to be:

$$\tilde{c}(s) = \frac{\tilde{a}(s) - \tilde{a}(s;T)}{1 - \tilde{a}(s;T)} \tag{37}$$

This distribution density is zero in the interval $[0, T)$, and thereafter exhibits the piecewise continuous behaviour of $b(t)$.

The mean inter-gap spacing is given by

$$\nu_c = \nu A^{-1}(T) \quad , \tag{38}$$

or, in other words, ν_c is just the limits of ν_q for large T. This is also the "intuitive" answer for the mean spacing between gaps. The variance of the inter-gap spacing is given by

$$\sigma_c^2 + \nu_c^2 = \left[\sigma^2 + \nu^2\right] A^{-1}(T) + 2\nu A^{-2}(T) \int_0^T t\, a(t)\, dt \tag{39}$$

where, as before, ν and σ^2 are the mean and variance of the mainstream headway distribution.

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* It is possible to obtain some limited results in the case where more than one car may enter; this topic will be the subject of a later report.

For the limiting cases of small and large values of critical gap, T, we obtain:

$$\lim_{T \to 0} \nu_c = \nu[1 + a_0 T + [\frac{a_1}{2} + a_0^2] T^2 + \ldots] \tag{40}$$

$$\lim_{T \to 0} \sigma_c^2 = \sigma^2 + (\sigma^2 - \nu^2) a_0 T + \left[a_0 \nu + \frac{a_1}{2}(\sigma^2 - \nu^2) + a_0^2 [\sigma^2 - 2\nu^2] \right] T^2 + \ldots \tag{41}$$

and

$$\lim_{T \to \infty} \nu_c = \nu A^{-1}(T) \tag{42}$$

$$\lim_{T \to \infty} \sigma_c^2 \approx (\sigma^2 + \nu^2) A^{-1}(T) + \nu^2 A^{-2}(T) \approx \nu^2 A^{-2}(T) \tag{43}$$

We notice that the coefficient of variation also approaches unity in limit, even though c(t) is not exponential.

In the case of Poisson traffic,

$$\mathcal{C}(s) = \frac{\lambda e^{-\lambda T} e^{-ST}}{S + \lambda e^{-\lambda T} e^{-ST}} \tag{44}$$

and we find:

$$\begin{array}{lll} c(t) = & 0 & 0 \le t < T \\ & \lambda e^{-\lambda T} \; [1] & T \le t \le \infty \\ & - \lambda e^{-2\lambda T} [\frac{\lambda(t-2T)}{1!}] & 2T \le t \le \infty \\ & + \lambda e^{-3\lambda T} [\frac{\lambda^2 (t-3T)^2}{2!}] & 3T \le t \le \infty \\ & - \lambda e^{-4\lambda T} [\frac{\lambda^3 (t-4T)^3}{3!}] & 4T \le t \le \infty \\ & \ldots & \ldots \end{array}$$

This distribution is shown in Figure 3, for $\lambda T = 1/2$, 1 and 2. One finds directly that:

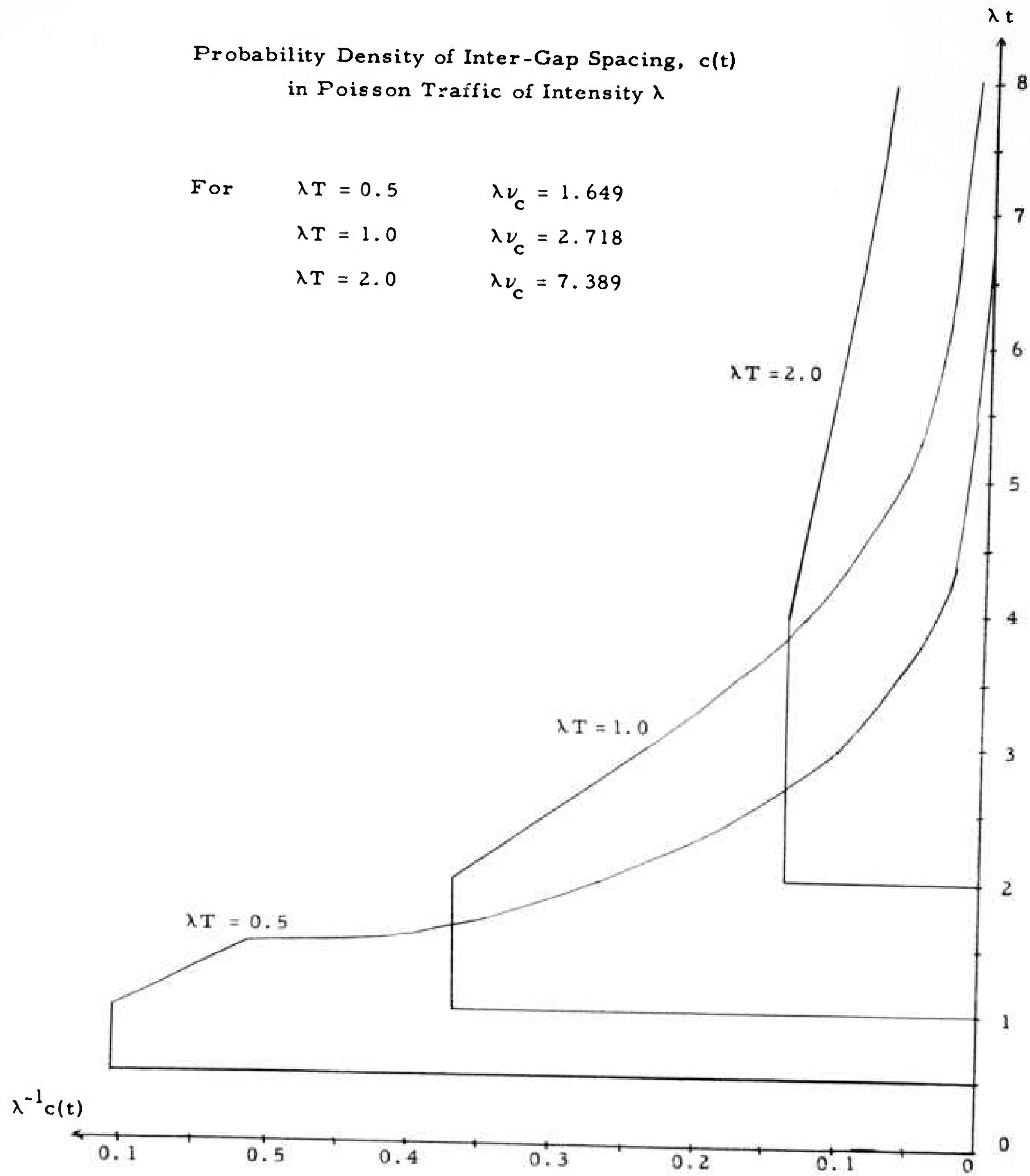
Probability Density of Inter-Gap Spacing, c(t)
in Poisson Traffic of Intensity λ
For λT = 0.5 λν_c = 1.649
λT = 1.0 λν_c = 2.718
λT = 2.0 λν_c = 7.389
λt
8
7
6
5
4
3
2
1
0
λT = 2.0
λT = 1.0
λT = 0.5
λ^-1c(t)
0.1
0.5
0.4
0.3
0.2
0.1
0

Figure 3

$$\nu_c = \lambda^{-1} e^{\lambda T} \tag{45}$$

$$\sigma_c^2 = \lambda^{-2} \left[e^{2\lambda T} - 2\lambda T\, e^{\lambda T} \right] . \tag{46}$$

The Counting Distribution of Gaps

We are now in a position to describe the emptying of a queue which is waiting on the secondary road to enter or cross the main stream of traffic. The wait of the first vehicle is a sample from the distribution $q(t)$, and if only one vehicle is allowed to enter the intersection per gap, successive vehicles have additional delays which are samples from the inter-gap distribution, $c(t)$

Thus, the probability $P_n(t)$ that exactly n cars from the secondary road enter the intersection is just the counting distribution for the passage of gaps. By a simple renewal argument,

$$P_0(t) = \int_t^{\infty} q(x)\, dx = Q(t) \tag{47}$$

$$P_1(t) = \int_0^{t} q(x)\, C(t-x)\, dx = q(t) * C(t)$$

$$P_2(t) = q(t) * C(t) * c(t)$$

$$\vdots \qquad\qquad \vdots$$

$$P_n(t) = q(t) * C(t) * c^{(n-1)*}(t)$$

or, in terms of the transforms,

$$\tilde{P}_0(s) = \frac{1 - \tilde{q}(s)}{s} \tag{48}$$

$$\tilde{P}_1(s) = \tilde{q}(s)\left[\frac{1-\tilde{c}(s)}{s}\right] \tag{48}$$

$$\tilde{P}_2(s) = \tilde{q}(s)\left[\frac{1-\tilde{c}(s)}{s}\right]\tilde{c}(s)$$

$$\vdots$$

$$\tilde{P}_n(s) = \tilde{q}(s)\left[\frac{1-\tilde{c}(s)}{s}\right][\tilde{c}(s)]^{n-1} \quad .$$

We can find the transforms of the first and second moments directly as:

$$\mathcal{L}[M(t)] = \frac{\tilde{q}(s)}{s[1-\tilde{c}(s)]} \tag{49}$$

$$\mathcal{L}[D(t) + M^2(t)] = \frac{\tilde{q}(s)[1+\tilde{c}(s)]}{s[1-\tilde{c}(s)]^2} \tag{50}$$

where M(t), and D(t) are the mean and variance of the counting distribution. Thus, after a choice of starting-up distribution, one has available the machinery to calculate the counting distribution of gaps. This is a long and tedious procedure for practically any headway distribution, however, and we shall be content with indicating a few asymptotic results.

First, it is known that in the limit of very large observation times, the state probabilities (47) approach normality [2][6]. It is also not difficult to show (via (49) and (50)), that

$$\lim_{t\to\infty} M(t) \approx \frac{t}{\nu_c} + \left[\frac{\sigma_c^2}{2\nu_c^2} + \frac{1}{2} - \frac{\nu_q}{\nu_c}\right] \approx \frac{t}{\nu_c} \tag{51}$$

$$\lim_{t\to\infty} D(t) \approx \left[\frac{\sigma_c^2}{\nu_c^2}\right]\frac{t}{\nu_c} + \text{constant} \tag{52}$$

so that, in the extreme limit, we find that the number of cars which are emptied from the secondary road is normally distributed, with mean and variance

$$\lim_{t \to \infty} M(t) \approx \frac{t}{\nu} A(T) \tag{53}$$

$$\lim_{t \to \infty} D(t) \approx k_c^2 \frac{t}{\nu} A(T) \tag{54}$$

We may interpret (53) as saying that the flow rate of gaps past the intersection is equal to the flow rate of cars in the main stream times the probability that a given headway is a large enough gap.

Other Gap Criteria*

In some applications, the simple criterion of "gap $\geq$ T" is not descriptive of actual behavior. Suppose, instead that we partition the headways into two sets G ('go'), and $\overline{G}$ ('don't go'), so that the definition of waiting time in Equation (1) becomes

$$\begin{aligned}
w &= 0 \quad \text{if} \quad t_0 \in G \\
w &= t_0 \quad \text{if} \quad t_0 \in \overline{G},\ t_1 \in G \\
w &= t_0 t t_1 \quad \text{if} \quad t_0,\ t_1 \in \overline{G},\ t_2 \in G \\
&\vdots \\
w &= \sum_{i=0}^{n} t_i \quad \text{if } t_0,\ t_1, \ldots t_n \in \overline{G},\ t_2 \in G
\end{aligned} \tag{55}$$

* This section was not part of the original paper [5].

A reappraisal of the development of the formulas for waiting-time distributions and their moments shows that this new gap criterion can be incorporated in our results by redefining all truncated distributions as:

$$f(t;T) = 0 \qquad t \in G \tag{56}$$
$$\qquad = f(t) \qquad t \in \overline{G}$$

and all the tail distributions as:

$$F(T) = \int_{t \in G} f(t)\, dt \tag{57}$$

With these interpretations, all of the formulas (8) - (16) and (36) - (39) are still valid.

Furthermore, it is possible to make the necessary modifications if the criterion of gap acceptance is a probabilistic one. Suppose that a gap of size t is "large enough" with probability $\alpha(t)$, and is not "large enough" with probability $1 - \alpha(t)$. Then these are just the probabilities of being in set G or $\overline{G}$, respectively, and the distribution which occur on the average are obtained by redefining

$$f(t;T) = f(t)\ [1 - \alpha(t)] \tag{58}$$

and

$$F(T) = \int_0^\infty \alpha(t)\ f(t)\, dt \quad . \tag{59}$$

With these interpretations, all of the formulas (8) - (16) and (36) - (39) hold, with (12) - (16) and (36) - (39) duplicating the results of Weiss and Maradulin [14].

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